Integrated Path Following and Collision Avoidance Using a Composite Vector Field

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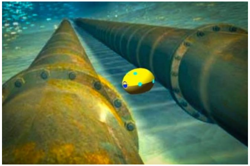
Introduction

Introduction

Path following is one of the fundamental capabilities for mobile robots.

(a) wheeled robots

(b) aerial robots



(c) underwater robots

Introduction

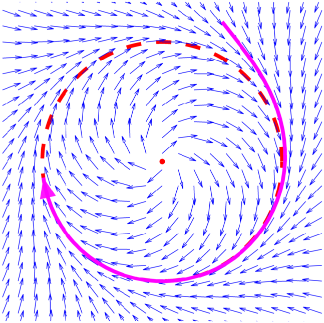
Path following is one of the fundamental capabilities for mobile robots.
But when there are many obstacles, **collision avoidance** is vital.

warehouse robots

Introduction

How to realize both capabilities in a unified theoretical framework?

Path following algorithms using a **vector field**: most accurate, least control effort (Sujit et al., 2014).



Vector field corresponding to the circle.

Introduction

Steps:

- Design a vector field for path following, and another one for collision avoidance.

We only consider the planar case now (i.e., $n = 2$).

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- Design a vector field for path following, and another one for collision avoidance.
- Combine two vector fields via bump functions.

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Introduction

Steps:

- Design a vector field for path following, and another one for collision avoidance.
- Combine two vector fields via bump functions.
- Obtaining the composite vector field $\chi : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^n$, we analyze the integral curves of it; i.e., the trajectories of $\dot{\xi}(t) = \chi(\xi(t))$.

We only consider the planar case now (i.e., $n = 2$).

Problem Formulation

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Desired path

$$\mathcal{P} = \{\xi \in \mathbb{R}^2 : \phi(\xi) = 0\},$$

where $\phi \in C^2 : \mathbb{R}^2 \to \mathbb{R}$.

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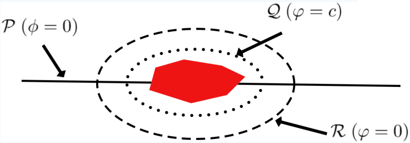
Obstacles, Reactive Boundary and Repulsive Boundary

$$\mathcal{O}_{\text{all}} = \{\mathcal{O}_i \subset \mathbb{R}^2 : i \in \mathcal{I}\}$$

$$\mathcal{R}_i = \{\xi \in \mathbb{R}^2 : \varphi_i(\xi) = 0\}$$

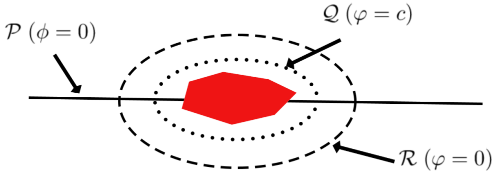
$$\mathcal{Q}_i = \{\xi \in \mathbb{R}^2 : \varphi_i(\xi) = c_i\},$$



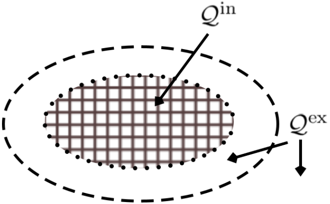
where $\mathcal{I} = \{1, 2, \ldots, m\}$, $c_i \neq 0$, $\varphi_i \in C^2 : \mathbb{R}^2 \to \mathbb{R}$.

It is assumed that $\mathcal{R}_i$ and $\mathcal{Q}_i$ are compact.

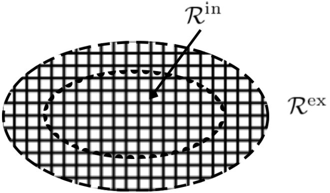
Problem Formulation



(a) Desired path $\mathcal{P}$, reactive boundary $\mathcal{R}$ and repulsive boundary $\mathcal{Q}$

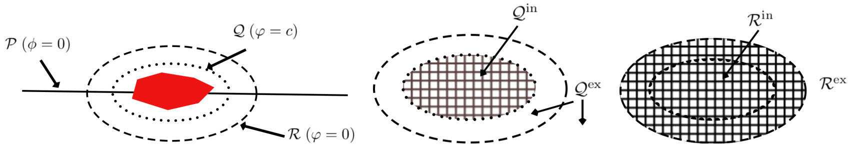


(b) Repulsive area $\mathcal{Q}^{\text{in}}$ and non-repulsive area $\mathcal{Q}^{\text{ex}}$



(c) Reactive area $\mathcal{R}^{\text{in}}$ and non-reactive area $\mathcal{R}^{\text{ex}}$

Assumption 1: For any $\xi_1, \xi_2 \in \mathbb{R}^2$, if $|\phi(\xi_1)| \leq |\phi(\xi_2)|$, then $\mathrm{dist}(\xi_1, \mathcal{P}) \leq \mathrm{dist}(\xi_2, \mathcal{P})$. (**absolute function value corresponds to distance**)



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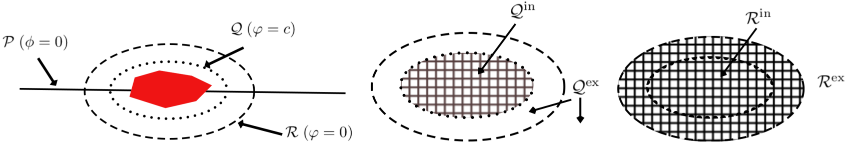
Assumption 2: $\mathcal{O}_i \subset \mathcal{Q}_i^{\mathrm{in}} \subset \mathcal{R}_i^{\mathrm{in}}$ and $\mathrm{dist}(\mathcal{Q}_i, \mathcal{R}_i) > 0$. **(obstacle enclosed by repulsive boundary, further enclosed by reactive boundary)**



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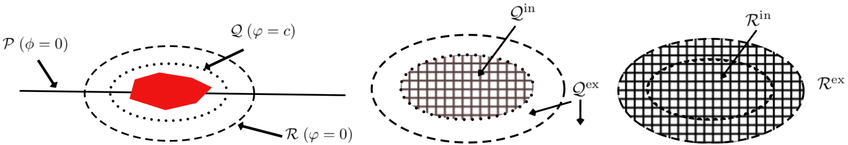
Assumption 3: $\mathcal{P} \not\subset \overline{\bigcup_{i \in \mathcal{I}} \mathcal{R}_i^{\mathrm{in}}}$. **(obstacles not occupy whole desired path)**



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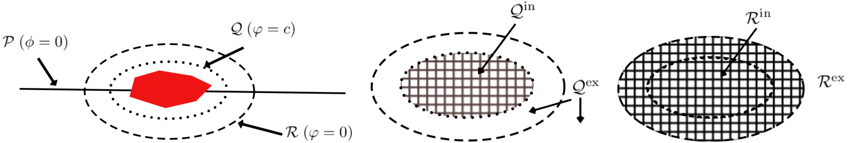
Assumption 4: For each $i \neq j \in \mathcal{I}$, $\mathrm{dist}(\mathcal{R}_i^{\mathrm{in}}, \mathcal{R}_j^{\mathrm{in}}) > 0$. **(obstacles not too close to each other)**



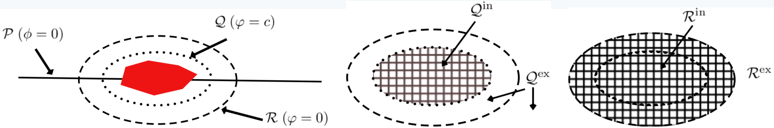
Problem Formulation

Vector **F**ield based integrated **C**ollision **A**voidance and **P**ath **F**ollowing (VF-CAPF)

Definition (VF-CAPF problem)

Design a continuously differentiable vector field $\chi : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}^2$ *for* $\dot{\xi}(t) = \chi(\xi(t))$ *satisfying*

1. *(**Path following**). If* $\mathcal{O}_{\text{all}} = \emptyset$*, the VF-PF problem is solved.*



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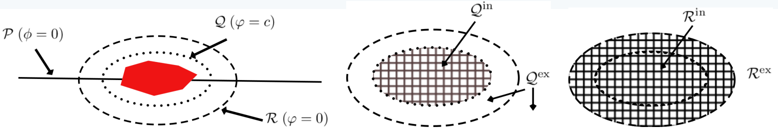
2. *(**Repulsive** $\overline{\mathcal{Q}^{\text{in}}}$). Trajectories do not enter closed repulsive area* $\overline{\mathcal{Q}^{\text{in}}}$*.*



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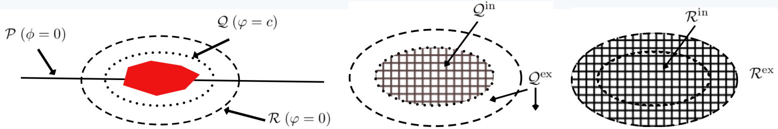
3. *(**Bounded error**). The path-following error* $\mathrm{dist}(\xi(t), \mathcal{P})$ *is bounded. In non-reactive areas, the path-following error is strictly decreasing.*



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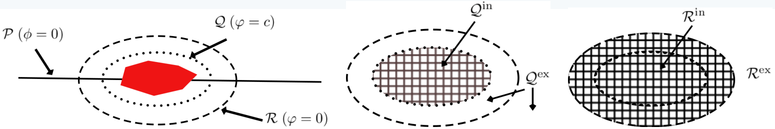
3. *(**Bounded error**). The path-following error* $\mathrm{dist}(\xi(t), \mathcal{P})$ *is bounded. In non-reactive areas, the path-following error is strictly decreasing.*
4. *(**Penetrable*** $\overline{\mathcal{R}_i^{\mathrm{in}}}$*). Whenever a trajectory is in the closed reactive area* $\overline{\mathcal{R}_i^{\mathrm{in}}}$*, there exists a future (finite) time instant when it is not in that area.*



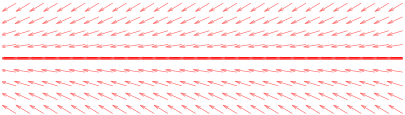
Animation: Single Obstacle

Composite VF Construction

Composite VF Construction

The vector fields $\chi_{\mathcal{P}}, \chi_{\mathcal{R}_i} : \mathbb{R}^2 \to \mathbb{R}^2$ associated with $\mathcal{P}$ and $\mathcal{R}_i$ are:

$$\chi_{\mathcal{P}}(\xi) = E \nabla \phi(\xi) - k_p \phi(\xi) \nabla \phi(\xi), \qquad \textit{PF vector field}$$



VF $\hat{\chi}_{\mathcal{P}}$ for desired path $\mathcal{P}$

Composite VF Construction

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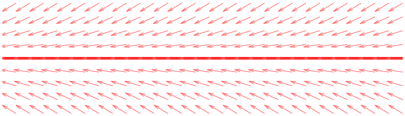
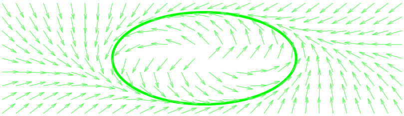
$$\chi_{\mathcal{P}}(\xi) = E\nabla\phi(\xi) - k_p\phi(\xi)\nabla\phi(\xi), \qquad \textit{PF vector field}$$

$$\chi_{\mathcal{R}_i}(\xi) = E\nabla\varphi_i(\xi) - k_{r_i}\varphi_i(\xi)\nabla\varphi_i(\xi), \qquad \textit{Reactive vector field}$$



VF $\hat{\chi}_{\mathcal{P}}$ for desired path $\mathcal{P}$



VF $\hat{\chi}_{\mathcal{R}}$ for reactive boundary $\mathcal{R}$

Composite VF Construction

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$$\chi_{\mathcal{P}}(\xi) = E\nabla\phi(\xi) - k_p\phi(\xi)\nabla\phi(\xi), \qquad \textit{PF vector field}$$
$$\chi_{\mathcal{R}_i}(\xi) = E\nabla\varphi_i(\xi) - k_{r_i}\varphi_i(\xi)\nabla\varphi_i(\xi), \qquad \textit{Reactive vector field}$$

The *critical set* $\mathcal{C}_{\mathcal{P}}$ and $\mathcal{C}_{\mathcal{R}_i}$ are

$$\mathcal{C}_{\mathcal{P}} = \{\xi \in \mathbb{R}^2 : \chi_{\mathcal{P}}(\xi) = 0\}, \quad \mathcal{C}_{\mathcal{R}_i} = \{\xi \in \mathbb{R}^2 : \chi_{\mathcal{R}_i}(\xi) = 0\},$$

It is assumed that $\mathrm{dist}(\mathcal{P}, \mathcal{C}_{\mathcal{P}}) > 0$ and $\mathrm{dist}(\mathcal{R}_i, \mathcal{C}_{\mathcal{R}_i}) > 0$.

Composite VF Construction

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It is assumed that $\mathrm{dist}(\mathcal{P}, \mathcal{C}_{\mathcal{P}}) > 0$ and $\mathrm{dist}(\mathcal{R}_i, \mathcal{C}_{\mathcal{R}_i}) > 0$.

The integral curves of $\chi_{\mathcal{P}}(\xi)$ either converge to $\mathcal{P}$ or $\mathcal{C}_{\mathcal{P}}$.

Similarly, the integral curves of $\chi_{\mathcal{R}_i}(\xi)$ either converge to $\mathcal{R}_i$ or $\mathcal{C}_{\mathcal{R}_i}$.
(Kapitanyuk, et al., 2017)

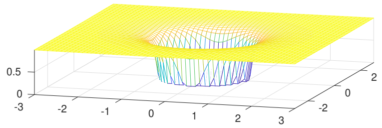
Composite VF Construction

Lemma 1 (bump functions)

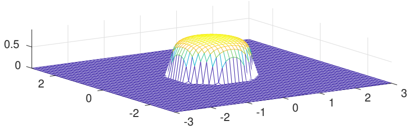
For $\mathcal{R}_i$ and $\mathcal{Q}_i$, there exist ***smooth*** *functions $\sqcup_{\mathcal{Q}_i}, \sqcap_{\mathcal{R}_i} : \mathbb{R}^2 \to [0, \infty]$:*

$$\sqcup_{\mathcal{Q}_i}(\xi) = \begin{cases} 0 & \xi \in \overline{\mathcal{Q}_i^{\text{in}}} \\ a_i(\xi) & \xi \in \mathcal{Q}_i^{\text{ex}}, \end{cases} \qquad \sqcap_{\mathcal{R}_i}(\xi) = \begin{cases} 0 & \xi \in \overline{\mathcal{R}_i^{\text{ex}}} \\ b_i(\xi) & \xi \in \mathcal{R}_i^{\text{in}}, \end{cases}$$

where $a_i : \mathcal{Q}_i^{\text{ex}} \subset \mathbb{R}^2 \to (0, \infty)$ and $b_i : \mathcal{R}_i^{\text{in}} \subset \mathbb{R}^2 \to (0, \infty)$ are bounded smooth functions.



(a) ***zero-inside*** *bump function* $\sqcup_{\mathcal{Q}_i}$ (zero values **inside** and **including** $\mathcal{Q}_i$)

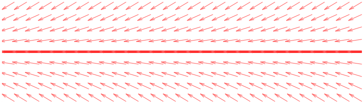


(b) ***zero-outside*** *bump function* $\sqcap_{\mathcal{R}_i}$ (zero values **outside** and **including** $\mathcal{R}_i$)

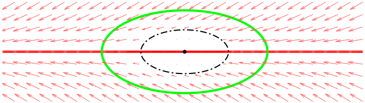
Composite VF Construction

Without loss of generality, consider only one obstacle.

$$\chi_c(\xi) = \sqcup_{\mathcal{Q}}(\xi)\hat{\chi}_{\mathcal{P}}(\xi)$$



(a) VF $\hat{\chi}_{\mathcal{P}}$ for desired path $\mathcal{P}$

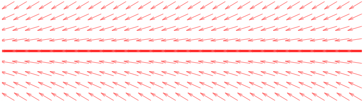


(b) VF $\sqcup_{\mathcal{Q}}(\xi)\hat{\chi}_{\mathcal{P}}(\xi)$

Composite VF Construction

Without loss of generality, consider only one obstacle.

$$\chi_c(\xi) = \sqcup_{\mathcal{Q}}(\xi)\hat{\chi}_{\mathcal{P}}(\xi) \quad \sqcap_{\mathcal{R}}(\xi)\hat{\chi}_{\mathcal{R}_i}(\xi)$$



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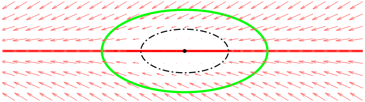
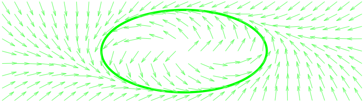
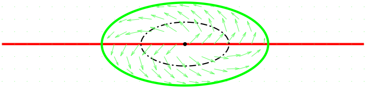


(b) VF $\sqcup_{\mathcal{Q}}(\xi)\hat{\chi}_{\mathcal{P}}(\xi)$



(c) VF $\hat{\chi}_{\mathcal{R}}$ for reactive boundary $\mathcal{R}$

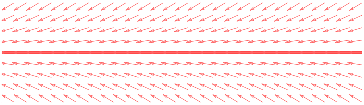


(d) VF $\sqcap_{\mathcal{R}}(\xi)\hat{\chi}_{\mathcal{R}}(\xi)$

Composite VF Construction

Without loss of generality, consider only one obstacle.

$$\chi_c(\xi) = \sqcup_{\mathcal{Q}}(\xi)\hat{\chi}_{\mathcal{P}}(\xi) + \sqcap_{\mathcal{R}}(\xi)\hat{\chi}_{\mathcal{R}_i}(\xi)$$



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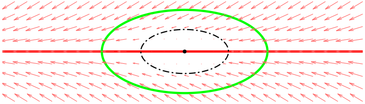
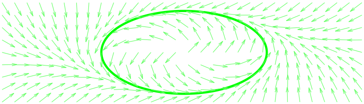
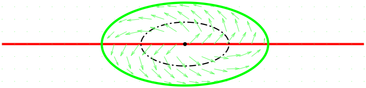


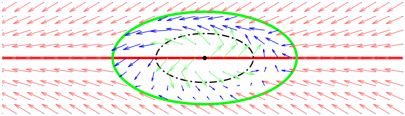
(b) VF $\sqcup_{\mathcal{Q}}(\xi)\hat{\chi}_{\mathcal{P}}(\xi)$



(c) VF $\hat{\chi}_{\mathcal{R}}$ for reactive boundary $\mathcal{R}$



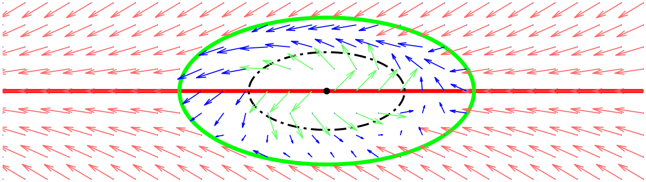
(d) VF $\sqcap_{\mathcal{R}}(\xi)\hat{\chi}_{\mathcal{R}}(\xi)$



(e) Composite VF $\chi_c(\xi)$

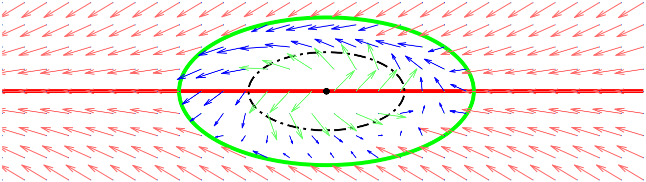
Composite VF Construction

Do the two vector fields cancel each other in the annulus area?



Main Results

The mixed area $\mathcal{M} = \mathcal{Q}^{\mathrm{ex}} \cap \mathcal{R}^{\mathrm{in}}$ will be investigated. By Nagumo's theorem, the closed mixed area $\overline{\mathcal{M}}$ is not positively invariant (not enough).



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Lemma 2 (fundamental limitation)

*If there are no critical points in the reactive area[2], **which is true for many practical cases**, then there is at least one **saddle point** of $\dot{\xi} = \chi_c(\xi)$ in the mixed area $\mathcal{M}$.*

[2] precisely, $\mathcal{C}_{\mathcal{P}} \cap \mathcal{R}^{\text{in}} = \emptyset$

Main Results

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Lemma 2 (fundamental limitation)

If there are no critical points in the reactive area[2], ***which is true for many practical cases***, *then there is at least one* ***saddle point*** *of* $\dot{\xi} = \chi_c(\xi)$ *in the mixed area* $\mathcal{M}$.

Remark 1

Note that if the condition is violated, it is possible that there are no equilibria in the mixed area $\mathcal{M}$, *and thus this limitation can be removed.*

[2] precisely, $\mathcal{C}_{\mathcal{P}} \cap \mathcal{R}^{\mathrm{in}} = \emptyset$

Main Results

Combining the previous results, the main theorem follows:

Theorem 1

The VF-CAPF problem is solved if the following conditions hold:

1. $\xi(0) \notin \mathcal{W}(\mathcal{C}_{\mathcal{P}})$, $\mathcal{W}(\mathcal{C}_{\mathcal{R}}) \cap \mathcal{Q} = \emptyset$, $\mathcal{C}_{\mathcal{P}}$ *is bounded;*
2. $\mathcal{C}_{\mathcal{P}} \cap \mathcal{R}^{\mathrm{in}} = \emptyset$ *and there is only one equilibrium* $c_0 \in \mathcal{C}_c$ *in the mixed area* $\mathcal{M}$*;*
3. *there exists a trajectory* $\xi(t)$ *starting from the repulsive boundary* $\mathcal{Q}$ *and reaching the reactive boundary* $\mathcal{R}$*.*

Example: Multiple Obstacles

$$\chi_c(\xi) = \prod_{i \in \mathcal{I}} \sqcup_{\mathcal{Q}_i}(\xi)\hat{\chi}_{\mathcal{P}}(\xi) + \sum_{i \in \mathcal{I}} \sqcap_{\mathcal{R}_i}(\xi)\hat{\chi}_{\mathcal{R}_i}(\xi),$$

Conclusion and Future Work

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Conclusion and Future Work

Future work

- moving obstacles;
- non-holonomic robot model; e.g, a unicycle model;

Appendix 0: Problem Formulation (rigorous version)

Definition (VF-CAPF problem)

1. *(Path following). If $\mathcal{O}_{\rm all} = \emptyset$, the VF-PF problem is solved.*
2. *(Repulsive $\overline{\mathcal{Q}^{\rm in}}$). If $\xi(0) \notin \overline{\mathcal{Q}_i^{\rm in}}$ for all $i \in \mathcal{I}$, then $\xi(t) \notin \overline{\mathcal{Q}_j^{\rm in}}$ for $t \geq 0$ and all $j \in \mathcal{I}$. If there exits $i \in \mathcal{I}$ such that $\xi(0) \in \overline{\mathcal{Q}_i^{\rm in}}$, then there exists $T > 0$, such that $\xi(t) \notin \overline{\mathcal{Q}_j^{\rm in}}$ for $t \geq T$ and all $j \in \mathcal{I}$.*
3. *(Bounded path-following error). There exists a positive finite constant M such that $\mathrm{dist}(\xi(t), \mathcal{P}) \leq M$ for $t \geq 0$. Moreover, for all nonempty connected time intervals $\Xi_j \subset \mathbb{R}$, $j \in \mathbb{N}$, such that $\xi(t) \notin \bigcup_i \overline{\mathcal{R}_i^{\rm in}}$ for $t \in \Xi_j$, the path-following error $\mathrm{dist}(\xi(t), \mathcal{P})$ is strictly decreasing on Ξ_j.*
4. *(Penetrable $\overline{\mathcal{R}_i^{\rm in}}$). Fixing $i \in \mathcal{I}$, if for almost all trajectories, there exists $t_0^e \in \mathbb{R}$ such that $\xi(t_0^e) \in \overline{\mathcal{R}_i^{\rm in}}$, then there exists $t_0^l > t_0^e$ such that $\xi(t_0^l) \notin \overline{\mathcal{R}_i^{\rm in}}$. In addition, the trajectory cannot cross the reactive boundary $\mathcal{R}_i$ infinitely fast‡.*

‡Suppose there exists a strictly increasing sequence of time instants $(t_i)_{i=1}^{\infty}$ such that a trajectory is in the exit set (Conley, 1978) of the reactive boundary at these instants; precisely, $\xi(t_i) \in \mathcal{R}^- := \{\xi_0 \in \mathcal{R} : \xi(0) = \xi_0, \forall \delta > 0, \xi([0, \delta)) \not\subset \mathcal{R}\}$. If $(t_i)_{i=1}^{\infty}$ is a Cauchy sequence, then the trajectory $\xi(t)$ is said to cross $\mathcal{R}$ infinitely fast.

Appendix I: Nagumo's theorem (Blanchini&Miani, 2008)

Definition 1 (Bouligand's tangent cone)

Given a closed set $\mathcal{S} \subset \mathbb{R}^n$, the tangent cone to $\mathcal{S}$ at $x \in \mathbb{R}^n$ is defined as follows:

$$\mathcal{T}_{\mathcal{S}}(x) = \{z \in \mathbb{R}^n : \liminf_{\tau \to 0} \frac{\mathrm{dist}(x + \tau z, \mathcal{S})}{\tau} = 0\}.$$

The tangent cone is nontrivial only on the boundary of $\mathcal{S}$.

Theorem 1 (Nagumo's theorem)

Consider the system $\dot{x}(t) = f(x(t))$ and assume that for each initial condition $x(0)$ in an open set $\mathcal{O}$ it admits a unique solution defined for all $t \geq 0$. Let $\mathcal{S} \subset \mathcal{O}$ be a closed set. Then, $\mathcal{S}$ is positively invariant for the system if and only if the velocity vector satisfies Nagumo's condition:

$$f(x) \in \mathcal{T}_{\mathcal{S}}(x), \quad \text{for all } x \in \partial\mathcal{S}.$$

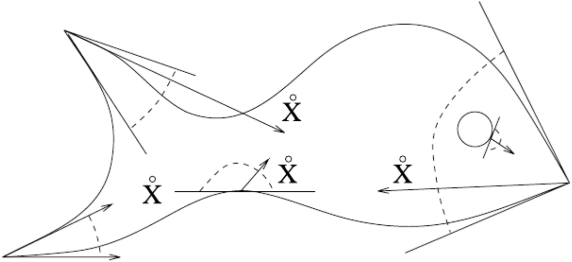

Fig. 4.1. Nagumo's conditions applied to a fish shaped set.

Appendix II: Index theorem (Khalil, 1996)

Consider the second-order autonomous system $\dot{x} = f(x)$, where $f(x) \in C^1$.

Poincaré index: Let C be a *simple closed curve* not passing through any equilibrium point. Consider the orientation of the vector field $f(x)$ at a point $p \in C$. Letting p traverse C in the *counterclockwise* direction, the vector $f(x)$ rotates continuously and, upon returning to the original position, must have rotated an angle $2k\pi$ for some integer k, where the angle is measured *counterclockwise*.

The integer is called the **index** of the closed curve C. If C is chosen to encircle a single isolated equilibrium point $\bar{x}$, then k is called the **index** of $\bar{x}$.

Appendix II

Theorem 2 (Index theorem)

1. *The index of a node, a focus, or a center is* $+1$.
2. *The index of a (hyperbolic) saddle is* -1.
3. *The index of a closed orbit is* $+1$.
4. ***The index of a closed curve not encircling any equilibrium point is*** 0.
5. *The index of a closed curve is equal to the sum of the indices of the equilibrium points within it.*

Appendix III: An example of no equilibria

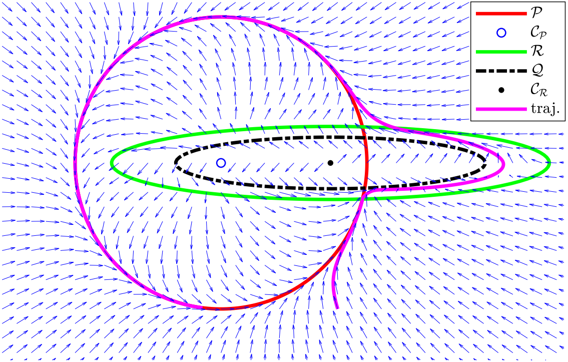

Figure 4: In this case, $\mathcal{C}_{\mathcal{P}} \cap \mathcal{R}^{\text{in}} \neq \emptyset$. There are no equilibrium points in the mixed area $\mathcal{M}$.

Appendix IV: Bump functions

The reactive boundary is described by a rotated ellipse in general:

$$\varphi(x,y) = \frac{((x-o_x)\cos\beta + (y-o_y)\sin\beta)^2}{a^2} + \frac{((x-o_x)\sin\beta - (y-o_y)\cos\beta)^2}{b^2} - 1 = 0$$

We choose the zero-inside bump function as

$$\sqcup_{\mathcal{Q}}(\xi) = \begin{cases} 0 & \xi \in \{\varphi(\xi) \leq c\} \\ \exp\left(\dfrac{l_1}{c-\varphi(\xi)}\right) & \xi \in \{\varphi(\xi) > c\} \end{cases} \tag{1}$$

and the zero-outside bump function as

$$\sqcap_{\mathcal{R}}(\xi) = \begin{cases} \exp\left(\dfrac{l_2}{\varphi(\xi)}\right) & \xi \in \{\varphi(\xi) < 0\} \\ 0 & \xi \in \{\varphi(\xi) \geq 0\}, \end{cases} \tag{2}$$

where $l_1 > 0$, $l_2 > 0$ are used to change the decaying or increasing rate of the bump functions.