Refining dichotomy convergence in vector-field guided path-following control

Weijia Yao\textsuperscript{1} Bohuan Lin\textsuperscript{1} Brian D. O. Anderson\textsuperscript{2} Ming Cao\textsuperscript{1}

2021 European Control Conference (ECC)

\textsuperscript{1}University of Groningen, the Netherlands
\textsuperscript{2}Australian National University, Australia
# Table of Contents

1. Introduction

2. Problem Formulation

3. Main Results

4. Conclusion
Introduction
The convergence of trajectories of a dynamical system to a **closed invariant set** is important in many control problems\(^1\).
The convergence of trajectories of a dynamical system to a **closed invariant set** is important in many control problems\(^1\).

The closed invariant set is described by the zero-level set of a continuous non-negative function \(f\). For convenience, \(f\) is referred to as the *level function* and its value at a point is called the point’s *level value*.

---

\(^1\)Kapitanyuk, et al., 2018; Yao, et al., 2020; Wang, et al., 2019; Qin, et al., 2018.
The convergence of trajectories of a dynamical system to a **closed invariant set** is important in many control problems\(^1\).

The closed invariant set is described by the zero-level set of a continuous non-negative function \(f\). For convenience, \(f\) is referred to as the *level function* and its value at a point is called the point’s *level value*.

One natural idea: use the level value along a trajectory to characterize the convergence to the zero-level set. But does this always work?

---

\(^1\)Kapitanyuk, et al., 2018; Yao, et al., 2020; Wang, et al., 2019; Qin, et al., 2018.
Another related question is: Does convergence to a closed invariant set imply convergence to a point in this set?

Another related question is: Does convergence to a closed invariant set imply convergence to a point in this set?

The answer is obvious: No. Then when will this implication hold?

\[\text{Absil, et al., SIOPT, 2005; Absil & Kurdyka, SCL, 2006.}\]
Introduction

Another related question is: Does convergence to a closed invariant set imply convergence to a point in this set?

The answer is obvious: No. Then when will this implication hold?

For gradient flows, this problem has been well studied\(^2\), but it is not completely clear for non-gradient flows.

Another related question is: Does convergence to a closed invariant set imply convergence to a point in this set?

The answer is obvious: No. Then when will this implication hold?

For gradient flows, this problem has been well studied\(^2\), but it is not completely clear for non-gradient flows.

We focus on one special kind of non-gradient flow given by a guiding vector field for path following.

Problem Formulation
Problem Formulation

Our work is motivated by the vector field guided path-following problem\(^3\),

**Desired path**

\[ \mathcal{P} = \{ \xi \in \mathbb{R}^n : \phi_i(\xi) = 0, i = 1, \ldots, n - 1 \}, \]

where \( \phi_i \in C^2 : \mathbb{R}^n \rightarrow \mathbb{R} \).

---

Our work is motivated by the vector field guided path-following problem\(^3\),

**Desired path**

\[
\mathcal{P} = \{\xi \in \mathbb{R}^n : \phi_i(\xi) = 0, i = 1, \ldots, n - 1\},
\]

where \(\phi_i \in C^2 : \mathbb{R}^n \to \mathbb{R}\).

Let \(f = \|(\phi_1, \ldots, \phi_{n-1})\|\), then \(\mathcal{P} = f^{-1}(0)\).

\(f\) is called the *level function*; for any point \(\xi \in \mathbb{R}^n\), the value \(f(\xi)\) is called the *level value*.

Problem Formulation

Our work is motivated by the vector field guided path-following problem\(^3\),

**Desired path**

\[ \mathcal{P} = \{\xi \in \mathbb{R}^n : \phi_i(\xi) = 0, i = 1, \ldots, n - 1\}, \]

where \( \phi_i \in C^2 : \mathbb{R}^n \to \mathbb{R} \).

Let \( f = \| (\phi_1, \ldots, \phi_{n-1}) \| \), then \( \mathcal{P} = f^{-1}(0) \).

\( f \) is called the *level function*; for any point \( \xi \in \mathbb{R}^n \), the value \( f(\xi) \) is called the *level value*.

Since \( f(\xi) = 0 \iff (\phi_1(\xi), \ldots, \phi_{n-1}(\xi)) = 0 \iff \xi \in \mathcal{P} \), one may use \( f(\xi) \) to quantify the distance from a point \( \xi \) to the desired path \( \mathcal{P} \).

The following question arises naturally:

Question 1 (Q1)

If \( f(\xi(t)) = \| (\phi_1(\xi(t)), \ldots, \phi_{n-1}(\xi(t))) \| \to 0 \) as \( t \to \infty \) along a continuous trajectory \( \xi(t) \) defined on \([0, \infty)\), is it true that the trajectory \( \xi(t) \) will converge to the set \( \mathcal{P} = f^{-1}(0) \)?
Question 1 (Q1)

If \( f(\xi(t)) = \| (\phi_1(\xi(t)), \ldots, \phi_{n-1}(\xi(t))) \| \to 0 \) as \( t \to \infty \) along a continuous trajectory \( \xi(t) \) defined on \([0, \infty)\), is it true that the trajectory \( \xi(t) \) will converge to the set \( \mathcal{P} = f^{-1}(0) \)?

Remark 1

Q1 does not depend on the path-following setting, but is relevant to any problem where the desired set is the zero-level set of a level function, and the convergence to the set is a requirement of the problem.
Problem Formulation

Question 1 (Q1)
If \( f(\xi(t)) = \| (\phi_1(\xi(t)), \ldots, \phi_{n-1}(\xi(t))) \| \to 0 \) as \( t \to \infty \) along a continuous trajectory \( \xi(t) \) defined on \([0, \infty)\), is it true that the trajectory \( \xi(t) \) will converge to the set \( \mathcal{P} = f^{-1}(0) \)?

Remark 1

**Q1** does not depend on the path-following setting, but is relevant to any problem where the desired set is the zero-level set of a level function, and the convergence to the set is a requirement of the problem.

However, the second question **Q2** to consider is closely related to the vector field guided path-following problem, as discussed below.
The guiding vector field $\chi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ for path following in $\mathbb{R}^2$ is:

$$\chi(\xi) = \underbrace{\text{Rot}(90^\circ) \nabla \phi(\xi)}_{\text{tangential/traversal}} - \underbrace{k\phi(\xi) \nabla \phi(\xi)}_{\text{orthogonal/converging}}, \quad (1)$$
The guiding vector field $\chi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ for path following in $\mathbb{R}^2$ is:

$$\chi(\xi) = \text{Rot}(90^\circ) \nabla \phi(\xi) - k\phi(\xi) \nabla \phi(\xi),$$

(1)

where $\text{Rot}(90^\circ)$ is a 90° rotation matrix, $\nabla \phi$ is the gradient of the function $\phi$, and $k > 0$ is a constant.

The guiding vector field $\chi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ in $\mathbb{R}^n$ for $n \geq 3$ is

$$\chi(\xi) = \perp \phi(\xi) - \sum_{i=1}^{n-1} k_i \phi_i(\xi) \nabla \phi_i(\xi),$$

(2)

where $\perp \phi$ is the wedge product of all the gradient vectors $\nabla \phi_i$ and $k_i > 0$ are constants for $i = 1, \ldots, n - 1$. 

$e(\xi) = (\phi_1(\xi), \ldots, \phi_{n-1}(\xi)) \in \mathbb{R}^{n-1}$. 

Hence, the level function $f = \|e\|$. 

\[\text{7/22}\]
Problem Formulation

The guiding vector field $\chi : \mathbb{R}^2 \to \mathbb{R}^2$ for path following in $\mathbb{R}^2$ is:

$$
\chi(\xi) = \underbrace{\text{Rot}(90^\circ) \nabla \phi(\xi)}_{\text{tangential/traversal}} - \underbrace{k \phi(\xi) \nabla \phi(\xi)}_{\text{orthogonal/converging}},
$$

(1)

The guiding vector field $\chi : \mathbb{R}^n \to \mathbb{R}^n$ in $\mathbb{R}^n$ for $n \geq 3$ is

$$
\chi(\xi) = \underbrace{\text{\perp}_\phi(\xi)}_{\text{tangential/traversal}} - \underbrace{\sum_{i=1}^{n-1} k_i \phi_i(\xi) \nabla \phi_i(\xi)}_{\text{orthogonal/converging}},
$$

(2)

where $\perp_\phi$ is the wedge product of all the gradient vectors $\nabla \phi_i$ and $k_i > 0$ are constants for $i = 1, \ldots, n-1$.

Let

$$
e(\xi) = (\phi_1(\xi), \ldots, \phi_{n-1}(\xi)) \in \mathbb{R}^{n-1}.
$$

(3)

Hence, the level function $f = \|e\|$. 


Problem Formulation

The integral curves of the guiding vector field (i.e., the trajectories of $\dot{x}(t) = \chi(x(t))$) converge to the desired path under some conditions. Trajectories may also converge to the singular set $C = \{x \in \mathbb{R}^n : \chi(x) = 0\}$, whose elements are called singular points. Under some mild assumptions, $P$ is an asymptotically stable limit cycle, and trajectories “spiral” and converge to the desired path but do not converge to any single point on the desired path. So how about the convergence to the singular set $C$?
Problem Formulation

The integral curves of the guiding vector field (i.e., the trajectories of \( \dot{\xi}(t) = \chi(\xi(t)) \)) converge to the desired path under some conditions. Trajectories may also converge to the singular set \( C \):

\[
C = \{ \xi \in \mathbb{R}^n : \chi(\xi) = 0 \},
\]

whose elements are called singular points.
Problem Formulation

The integral curves of the guiding vector field (i.e., the trajectories of \( \dot{\xi}(t) = \chi(\xi(t)) \)) converge to the desired path under some conditions.

Trajectories may also converge to the singular set \( C \):

\[
C = \{ \xi \in \mathbb{R}^n : \chi(\xi) = 0 \},
\]

whose elements are called *singular points*.

Under some mild assumptions, \( \mathcal{P} \) is an asymptotically stable limit cycle, and trajectories “spiral” and converge to the desired path but do **not** converge to any single point on the desired path.
The integral curves of the guiding vector field (i.e., the trajectories of \( \dot{\xi}(t) = \chi(\xi(t)) \)) converge to the desired path under some conditions. Trajectories may also converge to the singular set \( C \):

\[
C = \{ \xi \in \mathbb{R}^n : \chi(\xi) = 0 \},
\]

whose elements are called \textit{singular points}.

Under some mild assumptions, \( \mathcal{P} \) is an asymptotically stable limit cycle, and trajectories “spiral” and converge to the desired path but do \textbf{not} converge to any single point on the desired path.

So how about the convergence to the singular set \( C \)?
Question 2 (Q2)
When the trajectories converge to the singular set, will they converge to a singular point, or “spiral” towards the singular set and not converge to any single point of it?
Problem Formulation

Question 2 (Q2)
When the trajectories converge to the singular set, will they converge to a singular point, or “spiral” towards the singular set and not converge to any single point of it?
Main Results
Definition 1 (Metrical and topological convergence)

Consider a metric space \((M, d)\) and the topology induced by the metric \(d\). Suppose \(A \neq \emptyset\) is closed in \(M\), and let \((\xi_i)_{i=0}^{\infty} \in M\) be an infinite sequence of points. The sequence converges to \(A\) \textit{metrically} if for any \(\epsilon > 0\), there exists \(I > 0\) such that \(\text{dist}(\xi_i, A) \leq \epsilon\) for \(i \geq I\). It converges to \(A\) \textit{topologically} if for any open neighborhood \(^4\) \(U\) of \(A\), there exists \(I' > 0\) such that \(\xi_i(i \geq I') \subseteq U\).

\(^4\)An (open) neighborhood of \(A \subseteq M\) is an open set \(U \subseteq M\) such that \(A \subseteq U\).
Answer to Q1: Two convergence notions

**Definition 1 (Metrical and topological convergence)**

Consider a metric space \((M, d)\) and the topology induced by the metric \(d\). Suppose \(A \neq \emptyset\) is closed in \(M\), and let \((\xi_i)_{i=0}^{\infty} \in M\) be an infinite sequence of points. The sequence converges to \(A\) **metrically** if for any \(\epsilon > 0\), there exists \(I > 0\) such that \(\text{dist}(\xi_i, A) \leq \epsilon\) for \(i \geq I\). It converges to \(A\) **topologically** if for any open neighborhood \(U\) of \(A\), there exists \(I' > 0\) such that \(\xi_i(i \geq I') \subseteq U\).

**Remark 2**

Topological convergence is stronger than metrical convergence, while the former is relatively less studied in the control literature. This stronger notion is especially needed when a system evolves on some general topological space, or when a metric-independent convergence result is required.

\[\text{An (open) neighborhood of } A \subseteq M \text{ is an open set } U \subseteq M \text{ such that } A \subseteq U.\]
Answer to Q1: They are equivalent for a compact set

**Proposition 1**

Suppose $\mathcal{A} \neq \emptyset$ is *compact*. Then an infinite sequence of points converges metrically to the desired set $\mathcal{A} \iff$ it converges topologically to $\mathcal{A}$.

**Sketch of Proof.**

“$\iff$” is obvious.

“$\implies$”: Since $\mathcal{A} \neq \emptyset$ is compact, for any open neighborhood $U$ of $\mathcal{A}$, there exists an $\epsilon$-neighborhood\(^5\) $U_\epsilon$ of $\mathcal{A}$, such that $U_\epsilon \subseteq U$. $\Box$

---

\(^5\)An $\epsilon$-neighborhood $U_\epsilon$ of $\mathcal{A} \subseteq M$ is an open neighborhood of $\mathcal{A}$ defined by $U_\epsilon := \{ p \in M : \text{dist}(p, \mathcal{A}) < \epsilon \}$. 
Answer to Q1: They are equivalent for a compact set

**Proposition 1**

Suppose $\mathcal{A} \neq \emptyset$ is compact. Then an infinite sequence of points converges metrically to the desired set $\mathcal{A} \iff$ it converges topologically to $\mathcal{A}$.

**Sketch of Proof.**

“$\iff$” is obvious.

“$\Rightarrow$”: Since $\mathcal{A} \neq \emptyset$ is compact, for any open neighborhood $\mathcal{U}$ of $\mathcal{A}$, there exists an $\epsilon$-neighborhood $\mathcal{U}_\epsilon$ of $\mathcal{A}$, such that $\mathcal{U}_\epsilon \subseteq \mathcal{U}$. □

**Remark 3**

Usually, $\mathcal{A}$ is an equilibrium point, which is compact. Therefore, metrical convergence and topological convergence are equivalent.

---

$^5$An $\epsilon$-neighborhood $\mathcal{U}_\epsilon$ of $\mathcal{A} \subseteq \mathcal{M}$ is an open neighborhood of $\mathcal{A}$ defined by $\mathcal{U}_\epsilon := \{p \in \mathcal{M} : \text{dist}(p, \mathcal{A}) < \epsilon\}$. 
Answer to Q1: Local compactness

What if $A$ or $M$ is not compact? Then we use the one-point compactification of $M$ that is vital in subsequent proofs.

---

6The space $M$ is locally compact at $x \in M$ if there is a compact subspace $N \subseteq M$ that contains a neighborhood of $x$. If $M$ is locally compact at every point, then $M$ is said to be locally compact.
What if $A$ or $M$ is not compact? Then we use the one-point compactification of $M$ that is vital in subsequent proofs.

To ensure the one-point compactification of $M$ exists, we need:

**Assumption 1**

*The metric space $M$ is locally compact*\(^6\).

---

\(^6\)The space $M$ is *locally compact* at $x \in M$ if there is a compact subspace $N \subseteq M$ that contains a neighborhood of $x$. If $M$ is locally compact at every point, then $M$ is said to be locally compact.
What if $A$ or $M$ is not compact? Then we use the one-point compactification of $M$ that is vital in subsequent proofs.

To ensure the one-point compactification of $M$ exists, we need:

**Assumption 1**

*The metric space $M$ is locally compact*\(^6\).

This assumption is satisfied if $M$ is a smooth manifold or a Euclidean space $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Now we have the following theorem.

\(^6\)The space $M$ is *locally compact* at $x \in M$ if there is a compact subspace $N \subseteq M$ that contains a neighborhood of $x$. If $M$ is locally compact at every point, then $M$ is said to be locally compact.
Answer to Q1: Level value $\rightarrow 0 \iff$ convergence to the zero-level set

### Theorem 1

Let $\mathcal{A} := \{\xi \in \mathcal{M} : \|\phi(\xi)\| = 0\}$, where $\phi : \mathcal{M} \rightarrow \mathbb{R}^m$ is a continuous function. If $(\xi_i)_{i=0}^{\infty} \in \mathcal{M}$ is an infinite sequence of points such that $\|\phi(\xi_i)\| \rightarrow 0$ as $i \rightarrow \infty$, then the sequence converges topologically to the set $\mathcal{A} \cup \{\infty\}$ as $i \rightarrow \infty$. 
Answer to Q1: Level value $\rightarrow 0 \iff$ convergence to the zero-level set

**Theorem 1**

Let $A := \{\xi \in \mathcal{M} : \|\phi(\xi)\| = 0\}$, where $\phi : \mathcal{M} \to \mathbb{R}^m$ is a continuous function. If $(\xi_i)_{i=0}^\infty \in \mathcal{M}$ is an infinite sequence of points such that $\|\phi(\xi_i)\| \to 0$ as $i \to \infty$, then the sequence converges topologically to the set $A \cup \{\infty\}$ as $i \to \infty$.

**Sketch of proof.**

Consider the problem in the one-point compactification of $\mathcal{M}$. Namely, $\mathcal{M}$ can be embedded in a compact space $\mathcal{N}$, and $\infty$ is regarded as a particular point in $\mathcal{N}$. And then we prove by contradiction.
Answer to Q1: Level value $\rightarrow 0 \iff$ convergence to the zero-level set

**Theorem 1**

Let $\mathcal{A} := \{\xi \in \mathcal{M} : \|\phi(\xi)\| = 0\}$, where $\phi : \mathcal{M} \rightarrow \mathbb{R}^m$ is a continuous function. If $(\xi_i)_{i=0}^{\infty} \in \mathcal{M}$ is an infinite sequence of points such that $\|\phi(\xi_i)\| \rightarrow 0$ as $i \rightarrow \infty$, then the sequence converges **topologically** to the set $\mathcal{A} \cup \{\infty\}$ as $i \rightarrow \infty$.

**Remark 4**

Four mutually exclusive possibilities:

1. The sequence converges to $\mathcal{A}$;
2. The sequence converges to $\infty$;
3. The sequence converges to both $\mathcal{A}$ and $\infty$;
4. The sequence converges neither to $\mathcal{A}$ nor $\infty$. 

Answer to Q1: Level value $\rightarrow 0 \iff$ convergence to the zero-level set

**Theorem 1**

Let $\mathcal{A} := \{\xi \in \mathcal{M} : \|\phi(\xi)\| = 0\}$, where $\phi : \mathcal{M} \rightarrow \mathbb{R}^m$ is a continuous function. If $(\xi_i)_{i=0}^{\infty} \in \mathcal{M}$ is an infinite sequence of points such that $\|\phi(\xi_i)\| \rightarrow 0$ as $i \rightarrow \infty$, then the sequence converges topologically to the set $\mathcal{A} \cup \{\infty\}$ as $i \rightarrow \infty$.

**Remark 4**

Four mutually exclusive possibilities:

1. The sequence converges to $\mathcal{A}$;
2. The sequence converges to $\infty$;
3. The sequence converges to both $\mathcal{A}$ and $\infty$;
4. The sequence converges neither to $\mathcal{A}$ nor $\infty$.

However, if the set $\mathcal{A}$ is compact and a continuous trajectory is considered, then only the first two cases are possible.
Theorem 2

Let $\mathcal{A} := \{\xi \in \mathcal{M} : \|\phi(\xi)\| = 0\}$, where $\phi : \mathcal{M} \to \mathbb{R}^m$ is continuous. If $\mathcal{A}$ is compact, and $\xi : \mathbb{R}_{\geq 0} \to \mathcal{M}$ is continuous and $\|\phi(\xi(t))\| \to 0$ as $t \to \infty$, then $\xi(t)$ converges topologically to the set $\mathcal{A}$ or to $\infty$ exclusively as $t \to \infty$. 

---

Example 1

Suppose $\mathcal{A}$ is a unit circle (i.e. a compact desired path $\mathcal{P}$). The $\phi$ function is chosen as $\phi(x, y) = \left(x^2 + y^2 - 1\right) \exp(-x)$, and the vector field is constructed as before.
Answer to Q1: continued

**Theorem 2**

Let $A := \{ \xi \in \mathcal{M} : \|\phi(\xi)\| = 0 \}$, where $\phi : \mathcal{M} \to \mathbb{R}^m$ is continuous. If $A$ is compact, and $\xi : \mathbb{R}_{\geq 0} \to \mathcal{M}$ is continuous and $\|\phi(\xi(t))\| \to 0$ as $t \to \infty$, then $\xi(t)$ converges topologically to the set $A$ or to $\infty$ exclusively as $t \to \infty$.

**Example 1**

Suppose $A$ is a unit circle (i.e. a compact desired path $P$). The $\phi$ function is chosen as $\phi(x, y) = (x^2 + y^2 - 1)e^{-x}$, and the vector field is constructed as before.
Remark 5

Theorem 1 and 2 give a negative answer to Q1. If \( \mathcal{A} \) is compact, to exclude the possibility of trajectories escaping to infinity such that \( \|\phi(\xi_i)\| \to 0 \) implies topological convergence to \( \mathcal{A} \), one may retreat to:

1) Prove that trajectories are bounded. e.g., find a Lyapunov-like function \( V \) and a compact set \( \Omega_\alpha := \{x : V(x) \leq \alpha\} \), and prove that \( \dot{V} \leq 0 \) in this compact set \( \Omega_\alpha \).

2) Modify \( \phi(\cdot) \), if feasible, such that \( \|\phi(x)\| \) tends to a non-zero constant (possibly infinity) as \( \|x\| \) tends to infinity.

Regardless of whether the desired set \( \mathcal{A} \) is compact or not, one could impose an assumption introduced later.
**Lemma 1**

Consider two non-negative continuous functions $M_i : \mathcal{M} \rightarrow \mathbb{R}_{\geq 0}$, $i = 1, 2$. If for any given constant $\kappa > 0$, it holds that

$$\inf\{M_1(p) : M_2(p) \geq \kappa, p \in \mathcal{M}\} > 0,$$

(4)

then there holds $\lim_{k \to \infty} M_1(p_k) = 0 \implies \lim_{k \to \infty} M_2(p_k) = 0$, where $(p_k)_{k=1}^{\infty}$ is an infinite sequence of points in $\mathcal{M}$. 
**Answer to Q2: Convergence characterized by level functions**

**Lemma 1**

Consider two non-negative continuous functions \( M_i : M \to \mathbb{R}_{\geq 0}, \)
\( i = 1, 2. \) If for any given constant \( \kappa > 0, \) it holds that

\[
\inf\{ M_1(p) : M_2(p) \geq \kappa, p \in M \} > 0, \tag{4}
\]

then there holds \( \lim_{k \to \infty} M_1(p_k) = 0 \implies \lim_{k \to \infty} M_2(p_k) = 0, \) where \( (p_k)_{k=1}^{\infty} \) is an infinite sequence of points in \( M. \)

**Corollary 1**

Let \( A := \{ \xi \in M : \| \phi(\xi) \| = 0 \}, \) where \( \phi : M \to \mathbb{R}^m \) is a continuous function. Let \( M_1(\cdot) = \| \phi(\cdot) \| \) and \( M_2 = \text{dist}(\cdot, A) \) in Lemma 1, and suppose (4) holds. If \( (\xi_i)_{i=0}^{\infty} \) is a sequence of points \( \xi_i \in M \) such that \( \| \phi(\xi_i) \| \to 0 \) as \( i \to \infty, \) then the sequence converges metrically to \( A. \)
Corollary 1

Let $\mathcal{A} := \{\xi \in \mathcal{M} : \|\phi(\xi)\| = 0\}$, where $\phi : \mathcal{M} \to \mathbb{R}^m$ is a continuous function. Let $M_1(\cdot) = \|\phi(\cdot)\|$ and $M_2 = \text{dist}(\cdot, \mathcal{A})$ in Lemma 1, and suppose (4) holds. If $(\xi_i)_{i=0}^\infty$ is a sequence of points $\xi_i \in \mathcal{M}$ such that $\|\phi(\xi_i)\| \to 0$ as $i \to \infty$, then the sequence converges metrically to $\mathcal{A}$.

Remark 6

One can verify that the $\phi$ function in Example 1 does not satisfy the condition in (4) with $M_1$ and $M_2$ defined above, but the condition is met if the $\phi$ function is changed to $\phi(x, y) = x^2 + y^2 - 1$. 
According to the discussions above, we impose the following assumption:

**Assumption 2**

*For any given constant $\kappa > 0$, there holds*

$$\inf\{||e(\xi)|| : \xi \in \mathbb{R}^n, \text{dist}(\xi, P) \geq \kappa\} > 0,$$

*where $e(\cdot)$ is the path-following error vector.*
We will show that if a trajectory of $\dot{\xi}(t) = \chi(\xi(t))$ converges to the singular set $C$, then under some conditions, it converges to a point in $C$. This result depends on a property of real analytic functions:

**Lemma 1 (Łojasiewicz gradient inequality)**

Let $V: \mathbb{R}^n \to \mathbb{R}$ be a real analytic function on a neighborhood of $\xi^* \in \mathbb{R}^n$. Then there are constants $c > 0$ and $\mu \in [0, 1)$ such that

$$\|\nabla V(\xi)\| \geq c|V(\xi) - V(\xi^*)|^{\mu}$$

in some neighborhood $U$ of $\xi^*$.

**Theorem 3 (Refined Dichotomy Convergence)**

Let $\chi: \mathbb{R}^n \to \mathbb{R}^n$ be the guiding vector field for path following defined before. Suppose $\phi$ is real analytic and the singular set $C$ is bounded (hence compact). If a trajectory $\xi(t)$ of $\dot{\xi}(t) = \chi(\xi(t))$ converges metrically to the set $C$, then the trajectory converges to a point in $C$.

---

We will show that if a trajectory of $\dot{\xi}(t) = \chi(\xi(t))$ converges to the singular set $\mathcal{C}$, then under some conditions, it converges to a point in $\mathcal{C}$. This result depends on a property of real analytic functions:

**Lemma 1 (Łojasiewicz gradient inequality)**

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a real analytic function on a neighborhood of $\xi^* \in \mathbb{R}^n$. Then there are constants $c > 0$ and $\mu \in [0, 1)$ such that $\|\nabla V(\xi)\| \geq c|V(\xi) - V(\xi^*)|^{\mu}$ in some neighborhood $\mathcal{U}$ of $\xi^*$.

---

Answer to Q2: Refining dichotomy convergence

We will show that if a trajectory of $\dot{\xi}(t) = \chi(\xi(t))$ converges to the singular set $\mathcal{C}$, then under some conditions, it converges to a point in $\mathcal{C}$. This result depends on a property of real analytic functions:

**Lemma 1 (Łojasiewicz gradient inequality)**

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a real analytic function on a neighborhood of $\xi^* \in \mathbb{R}^n$. Then there are constants $c > 0$ and $\mu \in [0, 1)$ such that $\|\nabla V(\xi)\| \geq c|V(\xi) - V(\xi^*)|^{\mu}$ in some neighborhood $\mathcal{U}$ of $\xi^*$.

**Theorem 3 (Refined Dichotomy Convergence)**

Let $\chi : \mathbb{R}^n \to \mathbb{R}^n$ be the guiding vector field for path following defined before. Suppose $\phi$ is real analytic and the singular set $\mathcal{C}$ is bounded (hence compact). If a trajectory $\xi(t)$ of $\dot{\xi}(t) = \chi(\xi(t))$ converges metrically to the set $\mathcal{C}$, then the trajectory converges to a point in $\mathcal{C}$.

---

Answer to Q2: Examples

We show two simulation examples where the functions $\phi$ are real-analytic and non-real-analytic, respectively to verify Theorem 3.

**Example 2**

We choose a real-analytic $\phi$ function: $\phi(x, y) = x^3/3 - 9$, hence $\nabla \phi = (x^2, 0)^\top$. Therefore, $C$ is the y-axis, which is unbounded, and $\mathcal{P}$ is the vertical line $x = 3$. The vector field is $\chi(x, y) = x^2 (-k\phi(x, y), 1)^\top$, and the simulation results are shown below.
Example 3
We choose a non-real-analytic $\phi$ function. Consider a smooth but non-real-analytic function

$$b(x) = \begin{cases} \exp(1/x) & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases},$$

We can construct $\phi(x, y) = b(x) \left( x^3/3 - 9 \right)$. 
Conclusion
Conclusion and Future Work

We study the convergence with respect to a metric or a topology to a compact or non-compact desired set. The desired set is a zero-level set of a non-negative continuous level function.
Conclusion and Future Work

We study the convergence with respect to a metric or a topology to a compact or non-compact desired set. The desired set is a zero-level set of a non-negative continuous level function.

Main results:

- The convergence of the level value to zero does not necessarily imply the convergence of a continuous trajectory to the compact or non-compact desired set, while additional conditions or assumptions are provided to make this implication hold.
Conclusion and Future Work

We study the convergence with respect to a metric or a topology to a compact or non-compact desired set. The desired set is a zero-level set of a non-negative continuous level function.

Main results:

- The convergence of the level value to zero does not necessarily imply the convergence of a continuous trajectory to the compact or non-compact desired set, while additional conditions or assumptions are provided to make this implication hold.
- Real analyticity of the level function leads to the refined conclusion that convergence of a trajectory to a singular set implies convergence to a point in this set (i.e., limit cycles are precluded).
Thank you!