Refining dichotomy convergence in vector-field guided path-following control

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Introduction

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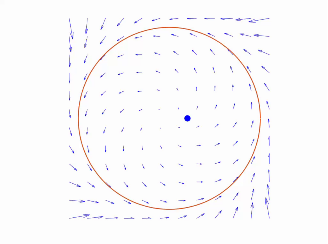

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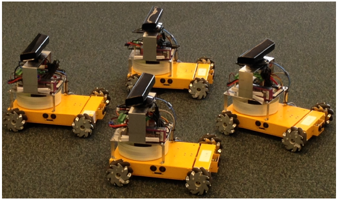

Formation maneuvering

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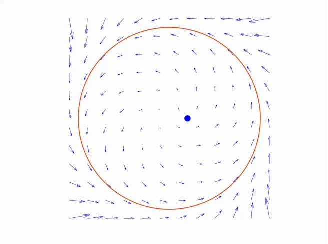

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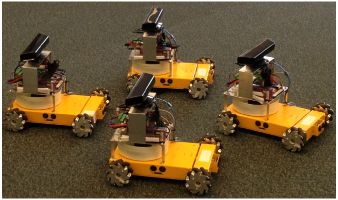

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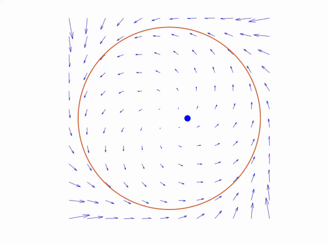
One natural idea: use the level value along a trajectory to characterize the convergence to the zero-level set. But does this always work?



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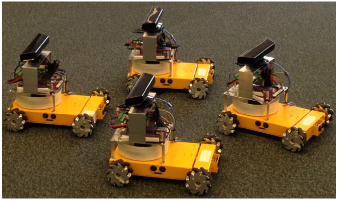

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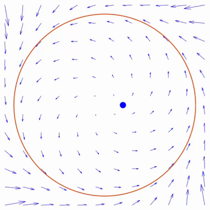
The answer is obvious: No. Then when will this implication hold?



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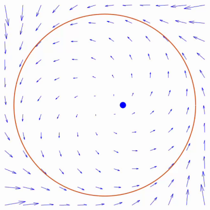

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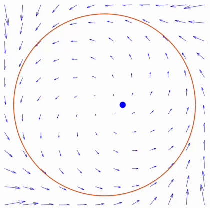
We focus on one special kind of non-gradient flow given by a guiding vector field for path following.



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Problem Formulation

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Our work is motivated by the vector field guided path-following problem[3],

Desired path

$$\mathcal{P} = \{\xi \in \mathbb{R}^n : \phi_i(\xi) = 0, i = 1, \dots, n-1\},$$

where $\phi_i \in C^2 : \mathbb{R}^n \to \mathbb{R}$.

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Let $f = \|(\phi_1, \ldots, \phi_{n-1})\|$, then $\mathcal{P} = f^{-1}(0)$.

f is called the *level function*; for any point $\xi \in \mathbb{R}^n$, the value $f(\xi)$ is called the *level value*.

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Since $f(\xi) = 0 \iff \big(\phi_1(\xi), \dots, \phi_{n-1}(\xi)\big) = 0 \iff \xi \in \mathcal{P}$, one may use $f(\xi)$ to quantify the distance from a point ξ to the desired path $\mathcal{P}$. The following question arises naturally:

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Question 1 (Q1)

If $f(\xi(t)) = \|(\phi_1(\xi(t)), \ldots, \phi_{n-1}(\xi(t)))\| \to 0$ as $t \to \infty$ along a continuous trajectory $\xi(t)$ defined on $[0, \infty)$, is it true that the trajectory $\xi(t)$ will converge to the set $\mathcal{P} = f^{-1}(0)$?

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Q1 *does not depend on the path-following setting, but is relevant to any problem where the desired set is the zero-level set of a level function, and the convergence to the set is a requirement of the problem.*

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However, the second question **Q2** to consider is closely related to the vector field guided path-following problem, as discussed below.

Problem Formulation

The guiding vector field $\chi : \mathbb{R}^2 \to \mathbb{R}^2$ for path following in $\mathbb{R}^2$ is:

$$\chi(\xi) = \underbrace{\mathrm{Rot}(90^\circ)\nabla\phi(\xi)}_{\text{tangential/traversal}} - \underbrace{k\phi(\xi)\nabla\phi(\xi)}_{\text{orthogonal/converging}}, \tag{1}$$

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where $\perp_\phi$ is the wedge product of all the gradient vectors $\nabla\phi_i$ and $k_i > 0$ are constants for $i = 1, \ldots, n-1$.

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Let

$$e(\xi) = \big(\phi_1(\xi), \ldots, \phi_{n-1}(\xi)\big) \in \mathbb{R}^{n-1}. \tag{3}$$

Hence, the level function $f = \|e\|$.

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Trajectories may also converge to the singular set $\mathcal{C}$:

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So how about the convergence to the singular set $\mathcal{C}$?

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Question 2 (Q2)

When the trajectories converge to the singular set, will they converge to a singular point, or "spiral" towards the singular set and not converge to any single point of it?

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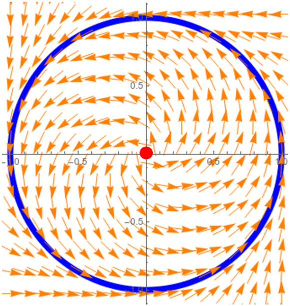

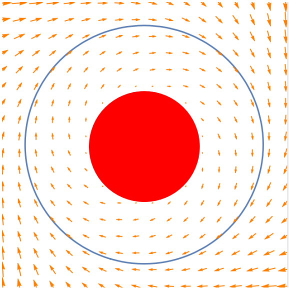

Main Results

Answer to Q1: Two convergence notions

Definition 1 (Metrical and topological convergence)

Consider a metric space $(\mathcal{M}, d)$ *and the topology induced by the metric* d*. Suppose* $\mathcal{A} \neq \emptyset$ *is closed in* $\mathcal{M}$*, and let* $(\xi_i)_{i=0}^{\infty} \in \mathcal{M}$ *be an infinite sequence of points. The sequence converges to* $\mathcal{A}$ *metrically if for any* $\epsilon > 0$*, there exists* $I > 0$ *such that* $\mathrm{dist}(\xi_i, \mathcal{A}) \leq \epsilon$ *for* $i \geq I$*. It converges to* $\mathcal{A}$ *topologically if for any open neighborhood*[4] $\mathcal{U}$ *of* $\mathcal{A}$*, there exists* $I' > 0$ *such that* $\xi_i (i \geq I') \subseteq \mathcal{U}$*.*

[4] An (open) neighborhood of $\mathcal{A} \subseteq \mathcal{M}$ is an open set $\mathcal{U} \subseteq \mathcal{M}$ such that $\mathcal{A} \subseteq \mathcal{U}$.

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Remark 2

Topological convergence is stronger than metrical convergence, while the former is relatively less studied in the control literature. This stronger notion is especially needed when a system evolves on some general topological space, or when a metric-independent convergence result is required.

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Answer to Q1: They are equivalent for a compact set

Proposition 1

Suppose $\mathcal{A} \neq \emptyset$ is compact. Then an infinite sequence of points converges metrically to the desired set $\mathcal{A}$ $\iff$ it converges topologically to $\mathcal{A}$.

Sketch of Proof.

" $\Longleftarrow$ " is obvious.

" $\Longrightarrow$ ": Since $\mathcal{A} \neq \emptyset$ is compact, for any open neighborhood $\mathcal{U}$ of $\mathcal{A}$, there exists an ϵ-neighborhood[5] $\mathcal{U}_\epsilon$ of $\mathcal{A}$, such that $\mathcal{U}_\epsilon \subseteq \mathcal{U}$. □

[5] An ϵ-neighborhood $\mathcal{U}_\epsilon$ of $\mathcal{A} \subseteq \mathcal{M}$ is an open neighborhood of $\mathcal{A}$ defined by $\mathcal{U}_\epsilon := \{p \in \mathcal{M} : \mathrm{dist}(p, \mathcal{A}) < \epsilon\}$.

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Remark 3

Usually, $\mathcal{A}$ is an equilibrium point, which is compact. Therefore, metrical convergence and topological convergence are equivalent.

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Answer to Q1: Local compactness

What if $\mathcal{A}$ or $\mathcal{M}$ is not compact? Then we use the *one-point compactification* of $\mathcal{M}$ that is vital in subsequent proofs.

[6] The space $\mathcal{M}$ is *locally compact* at $x \in \mathcal{M}$ if there is a compact subspace $\mathcal{N} \subseteq \mathcal{M}$ that contains a neighborhood of x. If $\mathcal{M}$ is locally compact at every point, then $\mathcal{M}$ is said to be locally compact.

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The metric space $\mathcal{M}$ is locally compact[6]*.*

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This assumption is satisfied if $\mathcal{M}$ is a smooth manifold or a Euclidean space $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Now we have the following theorem.

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Answer to Q1: Level value $\to 0 \not\Rightarrow$ convergence to the zero-level set

Theorem 1

Let $\mathcal{A} := \{\xi \in \mathcal{M} : \|\phi(\xi)\| = 0\}$, where $\phi : \mathcal{M} \to \mathbb{R}^m$ is a continuous function. If $(\xi_i)_{i=0}^{\infty} \in \mathcal{M}$ is an infinite sequence of points such that $\|\phi(\xi_i)\| \to 0$ as $i \to \infty$, then the sequence converges topologically to the set $\mathcal{A} \cup \{\infty\}$ as $i \to \infty$.

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Sketch of proof.

Consider the problem in the one-point compactification of $\mathcal{M}$. Namely, $\mathcal{M}$ can be embedded in a compact space $\mathcal{N}$, and ∞ is regarded as a particular point in $\mathcal{N}$. And then we prove by contradiction. $\square$

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Remark 4

Four mutually exclusive possibilities:

1. *The sequence converges to $\mathcal{A}$;*
2. *The sequence converges to ∞;*
3. *The sequence converges to both $\mathcal{A}$ and ∞;*
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However, if the set $\mathcal{A}$ is compact and a continuous trajectory is considered, then only the first two cases are possible.

Answer to Q1: continued

Theorem 2

Let $\mathcal{A} := \{\xi \in \mathcal{M} : \|\phi(\xi)\| = 0\}$, where $\phi : \mathcal{M} \to \mathbb{R}^m$ is continuous. If $\mathcal{A}$ is compact, and $\xi : \mathbb{R}_{\geq 0} \to \mathcal{M}$ is continuous and $\|\phi(\xi(t))\| \to 0$ as $t \to \infty$, then $\xi(t)$ converges topologically to the set $\mathcal{A}$ or to ∞ exclusively as $t \to \infty$.

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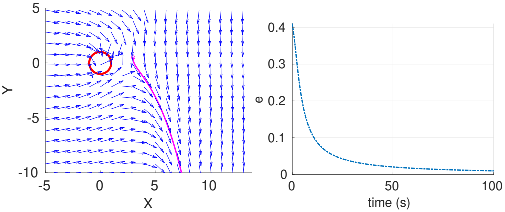
Example 1

Suppose $\mathcal{A}$ is a unit circle (i.e. a compact desired path $\mathcal{P}$). The ϕ function is chosen as $\phi(x, y) = (x^2 + y^2 - 1)\exp(-x)$, and the vector field is constructed as before.



Answer to Q1: Summary

Remark 5

*Theorem 1 and 2 give a negative answer to **Q1**. If $\mathcal{A}$ is compact, to exclude the possibility of trajectories escaping to infinity such that $\|\phi(\xi_i)\| \to 0$ implies topological convergence to $\mathcal{A}$, one may retreat to:*

1) Prove that trajectories are bounded. e.g., find a Lyapunov-like function V and a compact set $\Omega_\alpha := \{x : V(x) \le \alpha\}$, and prove that $\dot{V} \le 0$ in this compact set Ω_α.

2) Modify $\phi(\cdot)$, if feasible, such that $\|\phi(x)\|$ tends to a non-zero constant (possibly infinity) as $\|x\|$ tends to infinity.

Regardless of whether the desired set $\mathcal{A}$ is compact or not, one could impose an assumption introduced later.

Answer to Q2: Convergence characterized by level functions

Lemma 1

Consider two non-negative continuous functions $M_i : \mathcal{M} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$. *If for any given constant* $\kappa > 0$, *it holds that*

$$\inf\{M_1(p) : M_2(p) \geq \kappa, p \in \mathcal{M}\} > 0, \tag{4}$$

then there holds $\lim_{k\to\infty} M_1(p_k) = 0 \implies \lim_{k\to\infty} M_2(p_k) = 0$, *where* $(p_k)_{k=1}^{\infty}$ *is an infinite sequence of points in* $\mathcal{M}$.

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Corollary 1

Let $\mathcal{A} := \{\xi \in \mathcal{M} : \|\phi(\xi)\| = 0\}$, *where* $\phi : \mathcal{M} \to \mathbb{R}^m$ *is a continuous function. Let* $M_1(\cdot) = \|\phi(\cdot)\|$ *and* $M_2 = \mathrm{dist}(\cdot, \mathcal{A})$ *in Lemma 1, and suppose* (4) *holds. If* $(\xi_i)_{i=0}^{\infty}$ *is a sequence of points* $\xi_i \in \mathcal{M}$ *such that* $\|\phi(\xi_i)\| \to 0$ *as* $i \to \infty$, *then the sequence converges metrically to* $\mathcal{A}$.

Answer to Q2: Convergence characterized by level functions

Corollary 1

Let $\mathcal{A} := \{\xi \in \mathcal{M} : \|\phi(\xi)\| = 0\}$*, where* $\phi : \mathcal{M} \to \mathbb{R}^m$ *is a continuous function. Let* $M_1(\cdot) = \|\phi(\cdot)\|$ *and* $M_2 = \mathrm{dist}(\cdot, \mathcal{A})$ *in Lemma 1, and suppose* (4) *holds. If* $(\xi_i)_{i=0}^{\infty}$ *is a sequence of points* $\xi_i \in \mathcal{M}$ *such that* $\|\phi(\xi_i)\| \to 0$ *as* $i \to \infty$*, then the sequence converges metrically to* $\mathcal{A}$*.*

Remark 6

One can verify that the ϕ *function in Example 1 does not satisfy the condition in* (4) *with* M_1 *and* M_2 *defined above, but the condition is met if the* ϕ *function is changed to* $\phi(x, y) = x^2 + y^2 - 1$*.*

Answer to Q2: Assumptions

According to the discussions above, we impose the following assumption:

Assumption 2

For any given constant $\kappa > 0$*, there holds*

$$\inf\{||e(\xi)|| : \xi \in \mathbb{R}^n, \mathrm{dist}(\xi, \mathcal{P}) \geq \kappa\} > 0,$$

where $e(\cdot)$ *is the path-following error vector.*

Answer to Q2: Refining dichotomy convergence

We will show that if a trajectory of $\dot{\xi}(t) = \mathcal{X}(\xi(t))$ converges to the singular set $\mathcal{C}$, then under some conditions, it converges to a point in $\mathcal{C}$. This result depends on a property of real analytic functions:

[6] S. Łojasiewicz, "Sur la géométrie semi-et sous-analytique", 1993.

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Lemma 1 (Łojasiewicz gradient inequality)

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a real analytic function on a neighborhood of $\xi^* \in \mathbb{R}^n$. Then there are constants $c > 0$ and $\mu \in [0, 1)$ such that $\|\nabla V(\xi)\| \geq c|V(\xi) - V(\xi^*)|^{\mu}$ in some neighborhood $\mathcal{U}$ of ξ^*.

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Theorem 3 (Refined Dichotomy Convergence)

Let $\mathcal{X} : \mathbb{R}^n \to \mathbb{R}^n$ be the guiding vector field for path following defined before. Suppose ϕ is real analytic and the singular set $\mathcal{C}$ is bounded (hence compact). If a trajectory $\xi(t)$ of $\dot{\xi}(t) = \mathcal{X}(\xi(t))$ converges metrically to the set $\mathcal{C}$, then the trajectory converges to a point in $\mathcal{C}$.

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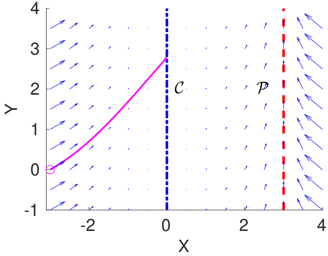
Answer to Q2: Examples

We show two simulation examples where the functions ϕ are real-analytic and non-real-analytic, respectively to verify Theorem 3.

Example 2

We choose a real-analytic ϕ function: $\phi(x, y) = x^3/3 - 9$, hence $\nabla\phi = (x^2, 0)^\top$. Therefore, $\mathcal{C}$ is the y-axis, which is unbounded, and $\mathcal{P}$ is the vertical line $x = 3$. The vector field is $\chi(x, y) = x^2\,(-k\phi(x, y), 1)^\top$, and the simulation results are shown below.



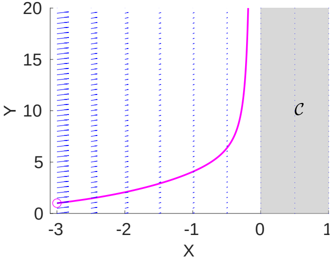
Answer to Q2: Examples

Example 3

We choose a non-real-analytic ϕ function. Consider a smooth but non-real-analytic function

$$b(x) = \begin{cases} \exp(1/x) & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases},$$

We can construct $\phi(x, y) = b(x)\left(x^3/3 - 9\right)$.



Conclusion

Conclusion and Future Work

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Main results:

- The convergence of the level value to zero does not necessarily imply the convergence of a continuous trajectory to the compact or non-compact desired set, while additional conditions or assumptions are provided to make this implication hold.
- Real analyticity of the level function leads to the refined conclusion that convergence of a trajectory to a singular set implies convergence to a point in this set (i.e., limit cycles are precluded).

Thank you!

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